

Asymptotic Behaviour of the Proper Length and Volume of the Schwarzschild Singularity

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Abstract

Though popular presentations give the Schwarzschild singularity as a point it is known that it is spacelike and not timelike. Thus it has a “length” and is not a “point”. In fact, its length must necessarily be infinite. It has been proved that the proper length of the Qadir-Wheeler suture model goes to infinity [1], while its proper volume shrinks to zero, and the asymptotic behaviour of the length and volume have been calculated. That model consists of two Friedmann sections connected by a Schwarzschild “suture”.

The question arises whether a similar analysis could provide the asymptotic behaviour of the Schwarzschild black hole near the singularity. It is proved here that, unlike the behaviour for the suture model, for the Schwarzschild essential singularity $\Delta s \sim K^{1/3} \ln K$ and $V \sim K^{-1} \ln K$, where K is the mean extrinsic curvature, or the York time.

1 Introduction

Despite popular presentations, it is well known [2] that the Schwarzschild (curvature) singularity is spacelike and hence is *not* a point. However, some of the implications of this fact have not been explicitly brought out and considered. If it is in effect, a line, it is natural to ask what its length is? The answer is intuitively obvious. Since it is singular it must have an infinite or a zero length. Further, since the singularity is not a point the length must be infinite. This answer raises two new questions. First, how is there a “collapse” if the length has gone infinite? Second, how does the length approach infinity as the singularity is approached? The answer to the former question is reasonably easy to see. It must be that the *volume* approaches zero even as the *length* approaches infinity. The second can not be answered without first agreeing on the choice of time slicing for the spacetime. Complete foliation of the Schwarzschild spacetime by spacelike hypersurfaces of constant mean extrinsic curvature, York or K -slices, had been obtained earlier [3] and then proved to be complete [4]. This paper will deal with these questions in more detail and particularly with the second question.

The question raised above had arisen earlier in the context of the “suture model” [5,

6] which modeled the formation of a black hole in a closed universe. This was done by attaching a slice of a denser closed Friedmann universe model to a rarer closed Friedmann model with a slice of equal mass cut out. Since the two sections evolve at different rates a matching at one instant will not be able to hold for any other instant. To keep them together, they are attached by a “suture” with a Schwarzschild geometry. It was required that the suture disappear at the big bang and evolve to “stitch” the two Friedmann sections together. Thus the denser Friedmann section could be regarded as a density perturbation in the rarer Friedmann section. At some stage the denser section collapses below the Schwarzschild radius, as seen from outside, and can therefore be regarded as a black hole. This model was sliced using the York time [2], i.e. by spacelike hypersurfaces of constant mean extrinsic curvature, K . The essential finding was that the black hole and big crunch singularities form simultaneously according to this slicing.

In the process of foliating the suture model, it was found that enormous amounts of computer time were required for obtaining the time-slices as one approached the big crunch. Since the Friedmann sections had collapsed to very small sizes, it would appear that it was the Schwarzschild suture that was causing the problem. However, there had been no clear understanding *why* this was happening. Subsequently, a heuristic argument was given which showed that the proper length of the hypersurface tends to infinity as the mean extrinsic curvature tends to zero [7]. This raised the question of whether there was, in fact, a collapse. By the same heuristic argument it was found that the volume of the slice does go to zero as required for collapse. A more rigorous analysis, by Hussain and Qadir (HQ) [1] showed that the basic expectations of the

heuristic argument, that the proper length diverges and the proper volume goes to zero, are correct (but the explicit form deduced heuristically turned out to be incorrect).

A naive analysis already shows the divergence of the proper length. The Schwarzschild metric written in the usual Schwarzschild coordinates,

$$ds^2 = -(1 - r_s/r)dt^2 + (1 - r_s/r)^{-1}dr^2 + r^2d\Omega^2, \quad (1)$$

is only valid for $r > r_s = 2Gm/c^2$. However, if we naively apply it for $r < r_s$ the dt^2 term gets a negative sign and the dr^2 term a positive sign. Thus r becomes a time parameter and t a space parameter. Thus, a time slice would be one in which we put r equal to a constant. Taking the slice at a given value of θ and φ we see that the proper length becomes

$$\Delta s = \sqrt{(r_s/r - 1)} \int dt, \quad (2)$$

over some appropriate range of integration. Thus $\Delta s \sim r^{-1/2}$. The problem is that the range of integration looks as if it should, itself be infinite. Further, if the range of integration were ignored, the volume, V , appears to go as $r^{3/2}$ and hence tends to 0. However, the range of integration casts doubt on whether the collapse genuinely occurs. The 0 may have to be multiplied by an ∞ to give a finite (or even infinite) value. As such, we need to calculate the behaviour more carefully.

In this paper we will demonstrate that the asymptotic behaviour of the Schwarzschild singularity follows the expectations arising from the suture model and the naive arguments given above. We will first perform the calculations in the compactified Kruskal

Szekres (CKS) coordinates [2, 8], used for constructing the Carter-Penrose (CP) diagram and then use the K-slice [3, 4, 9, 10] behaviour. We conclude with a summary and discussion.

2 Asymptotic Behaviour of Length of the Singularity in CKS Coordinates

The Schwarzschild metric in compactified Kruskal Szekres (CKS) coordinates (ψ, ξ) is given as [2, 8]

$$ds^2 = f^2(r) \frac{-d\psi^2 + d\xi^2}{4 \left(\cos^2 \frac{\psi}{2} - \sin^2 \frac{\xi}{2} \right)^2} + r^2 d\Omega^2, \quad (3)$$

where $-\pi/2 < \psi < \pi/2$ and $-\pi < \xi < \pi$,

$$f^2(r) = \frac{4r_s^3}{r} e^{-\frac{r}{2m}}, \quad (4)$$

and the radial parameter r and the CKS coordinates (ψ, ξ) are related by

$$\left(1 - \frac{r}{r_s}\right) \exp\left(\frac{r}{r_s}\right) = \tan\left(\frac{\psi + \xi}{2}\right) \tan\left(\frac{\psi - \xi}{2}\right). \quad (5)$$

To determine the proper length on the hypersurface, Δs ,

$$\Delta s = \int_{s_1}^{s_2} ds = \int_{s_1}^{s_2} \frac{f(r)}{2 \left(\cos^2 \frac{\psi}{2} - \sin^2 \frac{\xi}{2} \right)} \left[- \left(\frac{d\psi}{ds} \right)^2 + \left(\frac{d\xi}{ds} \right)^2 \right]^{\frac{1}{2}} ds, \quad (6)$$

where the ψ and ξ lie on the chosen hypersurface. Even though there is no geometrical significance to it, let us first try taking $\psi = \text{constt.}$ on the hypersurface, for the sake

of simplicity. Consider the hypersurface close to $\psi = \pi/2$. Thus, putting $\psi = \pi/2 - \epsilon$ (i.e. $r \rightarrow 0$), Eq.(6) becomes

$$\Delta s = \int_{-\pi/2-\epsilon}^{\pi/2+\epsilon} \frac{f(r)}{\sin \epsilon + \cos \xi} d\xi, \quad (7)$$

and $f^2(r)$ can be given as

$$f^2(r) = 4c^2 r_s^2 \epsilon^{-1/4} [1 + \epsilon^{1/2} + O(\epsilon)], \quad (8)$$

where c is a constant.

Using Eq.(8) in Eq.(7) and integrating we obtain

$$\Delta s = 2cr_s \epsilon^{-1/8} \ln \left(\frac{2}{\epsilon} \right) \left[1 + \frac{1}{2} \epsilon^{1/2} + O(\epsilon) \right]. \quad (9)$$

Eq.(9) shows that $\Delta s \sim -\epsilon^{-1/8} \ln(\epsilon)$. The constant, c , hides a problem. It depends on the spacelike coordinate ξ , in a secant function, which diverges as $\xi \rightarrow \pm\pi/2$. Thus the length has an extra infinite factor multiplying it. This problem can be resolved by going to the K , or York, slices, which have the added advantage of geometrical significance.

3 Asymptotic Behaviour of the Length of the Singularity Along the York Slice

The proper length of the singularity along the hypersurface of simultaneity, defined by York time, can be obtained from the analysis of HQ. The only modification to be made is that they used the boundaries of the Schwarzschild region in the suture model, so that on a given slice they took the spacelike coordinate ξ to lie between certain given ξ_1 and ξ_2 . For our present purpose $-\pi/2 \leq \xi \leq \pi/2$ at $r = 0$. HQ found that

$$\Delta s = 3^{5/6} \sqrt{2} r_s \ln \frac{\tan \xi_2 + \sec \xi_2}{\tan \xi_1 + \sec \xi_1} (K r_s)^{1/3} \times \left[1 + \frac{5}{4} 3^{-1/3} (K r_s)^{-2/3} + O(K r_s)^{-4/3} \right]. \quad (10)$$

Since $\sec \xi$ becomes infinite at $\pm\pi/2$, at first sight it appears that we need to take the limit as we approach these points. To this end we take $\xi_2 = \psi - 2\delta$, with $\psi = \pi/2 - \epsilon$, where δ and ϵ are small. Using these values in Eq.(5), in order to reach $\psi = \pi/2$ in the limits $\delta, \epsilon \rightarrow 0$, it is required that $\delta \geq \epsilon$ (for $\delta < \epsilon$ one reaches to the line $\psi = \xi$ first, which creates problems in reaching the singularity). It has been shown by HQ that ϵ goes to zero as r^2 and r goes to zero as $K^{-2/3}$, therefore ϵ goes to zero as $K^{-4/3}$.

Now taking $\xi_1 = 0$ and ξ_2 as mentioned above, and using the extreme value of δ i.e. $\delta = \epsilon$ in Eq.(10), we obtain

$$\Delta s = 3^{-1/6} 8 \sqrt{2} r_s (K r_s)^{1/3} \ln (K r_s) \times \left[1 + \frac{5}{4} 3^{-1/3} (K r_s)^{-2/3} + O(K r_s)^{-4/3} \right]. \quad (11)$$

The volume, V , is given by

$$V = 3^{-1/6} 8\sqrt{2}\pi r_s (Kr_s)^{-1} \ln(Kr_s) \times \left[1 + \frac{5}{4} 3^{-1/3} (Kr_s)^{-2/3} + O(Kr_s)^{-4/3} \right]. \quad (12)$$

Thus the basic results of HQ can be seen to hold here as well, but the proper length of the singularity goes to infinity even faster as there is an extra factor, $\ln(Kr_s)$, in Eq.(11). The significance of this factor is discussed in the conclusion. The second term in the brackets in Eqs.(11) and (12) becomes less than 1 for $Kr_s > 0.8$. Therefore the asymptotic behaviour given above is applicable for values of Kr_s much greater than 0.8. In the following section we present a foliation by K -slices upto this limit. There are problems of ill conditioning (excessive sensitivity) to initial values above 0.5, because of which we have not carried the calculation beyond the bare 0.8.

4 Foliation of the Schwarzschild spacetime by K -slices

As mentioned in the introduction, a complete K -slicing, had been obtained earlier. In this foliation it was required that the foliating hypersurfaces have zero slope at $\xi = 0$ and go to spacelike infinity. In order to find the length of the singularity, a more appropriate foliation would be one which stays inside the event horizon and in the limit coincides with the singularity at $r = 0$. For this purpose we require that the foliating hypersurfaces have zero slope at $\xi = 0$ and $\psi = (1 + \sigma)\xi$ ($\sigma \ll 1$). Using a procedure

similar to that adopted in [3, 4] we have obtain a foliation upto $Kr_s = 0.8$ for $\sigma = 0.1$ (see figure 1). Table 1 gives the length and the maximum height of the hypersurfaces for different values of Kr_s .

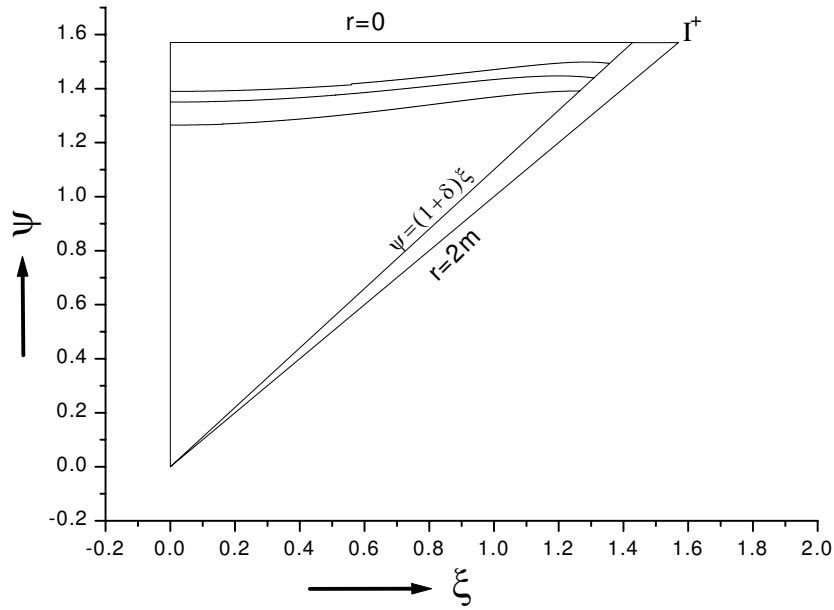


Figure 1: Spacelike hypersurfaces, of constant mean extrinsic curvature, in a Carter-Penrose diagram of the Schwarzschild geometry are shown. Only a few typical hypersurfaces are shown corresponding to $K = 0.1, 0.3$ and 0.8 .

Kr_s	r_i/r_s	$\psi_{initial}$	ψ_{final}	$length$
10^{-3}	.762	1.240	1.359	3.77
10^{-2}	.760	1.243	1.362	3.82
0.1	.745	1.265	1.391	4.18
0.2	.727	1.289	1.434	4.74
0.3	.7098	1.310	1.462	5.23
0.4	.687	1.336	1.485	5.78
0.5	.673	1.350	1.497	6.14
0.8	.630	1.390	1.500	6.59

Table 1: Eight K slices for different values of the mean extrinsic curvature, K , are described by the corresponding values for the initial value of r , r_i , the initial value of ψ in the Carter-Penrose diagram, $\psi_{initial}$, the final value, ψ_{final} , and the proper length of the slices.

5 Summary and Discussion

It was seen that the generic behaviour of the Schwarzschild singularity is that it has infinite proper length and occurs at an instant of time in the appropriate frame. (Of course, one can find frames in which there will be parts of the singularity that occur *before* and parts that occur *after* any given point on the singularity.) This fact appears even in the naive use of the usual Schwarzschild coordinates. It appears more clearly in the CKS coordinates used for the Carter-Penrose diagram. However, there is no

geometric significance to these coordinates. A frame in which the singularity *does* occur simultaneously is the one for which we use the York time, i.e. hypersurfaces of constant mean extrinsic curvature are the surfaces of simultaneity. It was shown that in this frame the proper length goes to infinity as the one third power of the York time, which goes to infinity at the singularity. That the collapse does, indeed, occur in this frame is shown by the fact that the volume shrinks to zero as the inverse two thirds power of the York time.

The interesting aspect of this observation is that the simple-minded view of the spherical collapse to a point needs to be modified. The collapse *is* to a point in some sense but is to a *line* in another. From outside the black hole it does indeed seem to collapse to a point. Using the picture of the black hole provided by the Schwarzschild coordinates inside, we see that the coefficient of the solid angle does shrink to zero and so the collapse is spherically symmetric. However, that coefficient is a timelike parameter. The point is that $r = 0$ is not *where* the collapse occurs but *when* it occurs. A new direction (along the t coordinate) has opened up inside the black hole, in some sense orthogonal to our usual space, on to which the matter collapses. As such, the volume does *not* shrink as the cube of r (which is a time) but as a lower power. The ambiguity of how to treat the asymptotic behaviour of the length, is taken care of by using the CKS coordinates. It was found that the proper length goes to infinity as $-\epsilon^{-1/8} \ln \epsilon$. Here $\epsilon \rightarrow 0$ as $r \rightarrow 0$. However r depends not only on ϵ but also on ξ , which measures a place on the singularity. In CKS coordinates, $r \sim 2r_s \epsilon^{1/2} (\sec \xi)^{1/2}$. Thus we see that $\Delta s \sim -r^{1/2} \ln r$ for $\xi \neq \pm\pi/2$. Consequently, the volume $V \sim r^{3/2} \ln r$ for these hypersurfaces, so long as we stay away from $\xi \neq \pm\pi/2$. The problem here is that we

need to include the points $\xi = \pm\pi/2$. This problem is avoided by considering York, or K -slices. It is found here that $\Delta s \sim K^{1/3} \ln K$ and $V \sim K^{-1} \ln K$, unlike the suture model in which the $\ln K$ is missing. Thus the length goes to infinity even faster than for the suture model.

It is of interest to also see what happens to the asymptotic behaviour of the length and the volume for $(n + 1)$ -dimensions. Generally $\Delta s \sim K^{1/3} \ln K$ still. However, $V \sim K^{1-2n/3} \ln K$. Thus for higher dimensional gravity the singularity becomes stronger. For $(2 + 1)$ - gravity $V \sim K^{-1/3} \ln K$, significantly slower than for the usual $(3 + 1)$ -gravity, and for $(1 + 1)$ -gravity $V \sim K^{1/3} \ln K$. Hence it *diverges* instead of going to zero! Thus, in 2-d gravity there would not be a meaningful *collapse* to a black hole singularity.

6 References

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